JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **211,** 386–394 (1997)
ARTICLE NO. AY975472

Integral Formulas and Inequalities for Rational Functions

Xin Li*

Department of Mathematics, University of Central Florida, Orlando, Florida 32816

Received January 17, 1996

Some Bernstein type inequalities using the integral norm are established for rational functions. A new proof of a Bernstein type inequality of Spijker is given as an application. © 1997 Academic Press

1. INTRODUCTION

One form of the classical Bernstein inequality for polynomials can be stated as

$$|P'(z)| \leq n\|P\|, \qquad |z| \leq 1, \tag{1}$$

for every $P \in \mathcal{P}_n$, the set of algebraic polynomials of degree at most n, where $\|\cdot\|$ denotes the sup norm on the unit circle $\mathbf{T} := \{z : |z| = 1\}$. Although the generalization of the Bernstein inequality to rational functions has been obtained and used in the study of rational approximation [PP], only recently have the sharp forms of such type inequalities been established independently in [BE, LMR]. Among the various forms, we mention the following two results: Let $\alpha_k \in \mathbf{C} \setminus \mathbf{T}$, $k = 1, 2, \ldots, n$, be given and let $w_n(z) := \prod_{k=1}^n (1 - \bar{\alpha}_k z)$,

$$B_-(z) := \prod_{|\alpha_k|<1} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \qquad B_+(z) := \prod_{|\alpha_k|>1} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z},$$

*Research supported, in part, by the University of Central Florida In-house Grant 94-0711.
E-mail address: xli@pegasus.cc.ucf.edu.

0022-247X/97 $25.00

and $B_n(z) := B_-(z)B_+(z)$ (as usual, $\prod_\varnothing := 1$). Define

$$\mathbf{R}_n := \left\{ \frac{p(z)}{w_n(x)} : p \in \mathbf{P}_n \right\}. \tag{2}$$

THEOREM 1 [BE, Theorem 1]. *For $r \in \mathbf{R}_n$ and $z \in \mathbf{T}$, we have*

$$|r'(z)| \le \max\left\{ \sum_{|\alpha_k|<1} \frac{1-|\alpha_k|^2}{|z-\alpha_k|^2}, \sum_{|\alpha_k|>1} \frac{|\alpha_k|^2-1}{|z-\alpha_k|^2} \right\} \|r\|.$$

If the first sum is not less than the second sum for a fixed $z \in \mathbf{T}$, then the equality holds for $r(z) = cB_-(z)$, $c \in \mathbf{C}$; otherwise, the equality holds for $r(z) = cB_+(z)$, $c \in \mathbf{C}$.

THEOREM 2 [LMR, Theorem 2]. *Assume $|\alpha_k| < 1$, $k = 1, 2, \dots, n$. Then, for $r \in \mathbf{R}_n$ and $z \in \mathbf{T}$,*

$$|r'(z)| + |r^{*\prime}(z)| \le \sum_{k=1}^{n} \frac{1-|\alpha_k|^2}{|z-\alpha_k|^2} \|r\|,$$

where $r^*(z) = B_n(z)\overline{r(1/\bar{z})}$. The equality holds for $r(z) = tB_n(z)$, $t \in \mathbf{T}$.

It is easy to see that either Theorem 1 or Theorem 2 implies that, if $|\alpha_k| < 1$, $k = 1, 2, \dots, n$, then, for $r \in \mathbf{R}_n$ and $z \in \mathbf{T}$,

$$|r'(z)| \le |B_n'(z)|\, \|r\|,$$

which is a natural generalization of the Bernstein inequality for polynomials to rational functions: when $\alpha_k = 0$, $k = 1, 2, \dots, n$, we have $B_n(z) = z^n$ and so the above inequality reduces to (1). The proofs in [BE, LMR] are both related to some interpolation formulas though they appear to be totally different. In this note, we use a new approach to establish a version of the Bernstein type inequality using the integral norm. As an application, we will give an alternative proof of the following inequality proved by Spijker [S] in 1991: If $r(z) = p(z)/q(z)$, where $p(z)$, $q(z) \in \mathbf{P}_n$ with $q(z) \ne 0$ on $\mathbf{T}$, then

$$\int_{\mathbf{T}} |r'(z)|\, |dz| \le 2\pi n \|r\|. \tag{3}$$

This inequality was first conjectured by LeVeque and Trefethen [LT] in 1984. For the application in numerical analysis, generalization, and related historical remarks concerning this result, see [WT].

2. THE INTEGRAL FORMULAS

Let $|\alpha_k| < 1$ for $k = 1, 2, \ldots, m$, and $|\alpha_k| > 1$ for $k = m+1, \ldots, n$. Then the two Blaschke products B_- and B_+ become

$$B_-(z) = \prod_{k=1}^{m} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \qquad \text{and} \qquad B_+(z) = \prod_{k=m+1}^{n} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}.$$

One of our main results is the following integral formula for the derivative of a rational function with poles off the unit circle $\mathbf{T}$.

LEMMA 3. *If $r \in \boldsymbol{R}_n$ and $|z| \le 1$ then*

$$r'(z) = \frac{1}{2\pi i} \int_{\mathbf{T}} r(\zeta)\overline{\zeta} \left\{ \left[\frac{1 - B_-(z)\overline{B_-(\zeta)}}{1 - z\overline{\zeta}} \right]^2 - \left[\frac{1 - B_+(z)\overline{B_+(\zeta)}}{1 - z\overline{\zeta}} \right]^2 \right\} |d\zeta|.$$

Proof. First, let us assume that the α_k's are nonzero and distinct. Define $w_-(z) = \prod_{k=1}^{m}(1 - \overline{\alpha}_k z)$ and $w_+(z) = \prod_{k=m+1}^{n}(1 - \overline{\alpha}_k z)$. Then, every $r \in \boldsymbol{R}_n$ has only simple poles and can be decomposed into

$$r(z) = \frac{p_1(z)}{w_-(z)} + \frac{p_2(z)}{w_+(z)} =: r_1(z) + r_2(z)$$

with $p_1 \in \boldsymbol{P}_m$ and $p_2 \in \boldsymbol{P}_{n-m-1}$.

It is easy to verify that

$$r_1(z) = r_1(\infty) + \sum_{k=1}^{m} \frac{p_1(1/\overline{\alpha}_k)}{w'_-(1/\overline{\alpha}_k)} \frac{-\overline{\alpha}_k}{1 - \overline{\alpha}_k z}.$$

So,

$$r'_1(z) = \sum_{k=1}^{m} \frac{p_1(1/\overline{\alpha}_k)}{w'_-(1/\overline{\alpha}_k)} \frac{-\overline{\alpha}_k^2}{(1 - \overline{\alpha}_k z)^2}.$$

Now, applying the residue theorem (for the outside of $\mathbf{T}$) and then using the above equality, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{r_1(\zeta)}{\zeta^2} \left[\frac{1 - B_-(z)/B_-(\zeta)}{1 - z/\zeta} \right]^2 d\zeta \\ = \sum_{k=1}^{m} \frac{p_1(1/\overline{\alpha}_k)}{w'_-(1/\overline{\alpha}_k)} \frac{-\overline{\alpha}_k^2}{(1 - \overline{\alpha}_k z)^2} = r'_1(z). \end{aligned}$$

Similarly, for r_2 we have

$$\frac{1}{2\pi i}\int_{\mathbf{T}} \frac{r_2(\zeta)}{\zeta^2}\left[\frac{1-B_+(z)/B_+(\zeta)}{1-z/\zeta}\right]^2 d\zeta$$
$$= \sum_{k=m+1}^{n} \frac{p_2(1/\overline{\alpha}_k)}{w'_+(1/\overline{\alpha}_k)} \frac{\overline{\alpha}_k^2}{(1-\overline{\alpha}_k z)^2} = -r_2'(z).$$

Therefore,

$$\begin{aligned} r'(z) &= r_1'(z) + r_2'(z) \\ &= \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{r_1(\zeta)}{\zeta^2}\left[\frac{1-B_-(z)/B_-(\zeta)}{1-z/\zeta}\right]^2 d\zeta \\ &\quad - \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{r_2(\zeta)}{\zeta^2}\left[\frac{1-B_+(z)/B_+(\zeta)}{1-z/\zeta}\right]^2 d\zeta. \end{aligned} \tag{4}$$

Note that

$$\int_{\mathbf{T}} \frac{r_2(\zeta)}{\zeta^2}\left[\frac{1-B_-(z)/B_-(\zeta)}{1-z/\zeta}\right]^2 d\zeta = 0 \tag{5}$$

since the integrand is analytic on and outside $\mathbf{T}$ and is of order $O(\zeta^{-2})$ as $\zeta \to \infty$. Similarly

$$\int_{\mathbf{T}} \frac{r_1(\zeta)}{\zeta^2}\left[\frac{1-B_+(z)/B_+(\zeta)}{1-z/\zeta}\right]^2 d\zeta = 0 \tag{6}$$

because the integrand is analytic on and inside $\mathbf{T}$. So, the two integrals in (4) can be combined into one to yield

$$\begin{aligned} r'(z) = \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{r(\zeta)}{\zeta^2}\Bigg\{&\left[\frac{1-B_-(z)/B_-(\zeta)}{1-z/\zeta}\right]^2 \\ &-\left[\frac{1-B_+(z)/B_+(\zeta)}{1-z/\zeta}\right]^2\Bigg\} d\zeta. \end{aligned}$$

This establishes the integral formula in the lemma when all α_k are nonzero and distinct. The general case can be obtained from this case by a limit procedure as both sides of the formula are continuous functions of α_k in $\mathbf{C} \setminus \mathbf{T}$. ∎

Remark. The integral formula is a generalization of Lemma 4.3 in [JLMR] which corresponds to the case when $m = n$.

LEMMA 4. *For $z \in \mathbf{T}$,*

$$\frac{1}{2\pi}\int_{\mathbf{T}}\left\{\left|\frac{1-B_-(z)\overline{B_-(\zeta)}}{1-z\bar{\zeta}}\right|^2+\left|\frac{1-B_+(z)\overline{B_+(\zeta)}}{1-z\bar{\zeta}}\right|^2\right\}|d\zeta| = |B'_\#(z)|,$$

where $B_\#(z) := B_-(z)/B_+(z)$.

Proof. For $w \in \mathbf{T}$, define

$$r_w(z) = w\overline{B_-(w)}B_-(z) - w\overline{B_+(w)}B_+(z).$$

Then $r_w \in \mathcal{R}_n$. So, using Lemma 3 together with relations (5) and (6), we get

$$r'_w(z) = \frac{1}{2\pi i}\int_{\mathbf{T}}\frac{w\overline{B_-(w)}B_-(\zeta)}{\zeta^2}\left[\frac{1-B_-(z)\overline{B_-(\zeta)}}{1-z\bar{\zeta}}\right]^2 d\zeta$$
$$+\frac{1}{2\pi i}\int_{\mathbf{T}}\frac{w\overline{B_+(w)}B_+(\zeta)}{\zeta^2}\left[\frac{1-B_+(z)\overline{B_+(\zeta)}}{1-z\bar{\zeta}}\right]^2 d\zeta. \quad (7)$$

Note that $(1-u)^2 = -u|1-u|^2$ for $|u| = 1$, so, for $z, \zeta \in \mathbf{T}$,

$$\left[\frac{1-B_\pm(z)\overline{B_\pm(\zeta)}}{1-z\bar{\zeta}}\right]^2 = \frac{B_\pm(z)\overline{B_\pm(\zeta)}}{z\bar{\zeta}}\left|\frac{1-B_\pm(z)\overline{B_\pm(\zeta)}}{1-z\bar{\zeta}}\right|^2.$$

Using this in (7) and letting $w = z$ give us

$$\left|\frac{B'_-(z)}{B_-(z)}-\frac{B'_+(z)}{B_+(z)}\right|$$
$$= \frac{1}{2\pi}\int_{\mathbf{T}}\left\{\left|\frac{1-B_-(z)\overline{B_-(\zeta)}}{1-z\bar{\zeta}}\right|^2+\left|\frac{1-B_+(z)\overline{B_+(\zeta)}}{1-z\bar{\zeta}}\right|^2\right\}|d\zeta|,$$

which implies the formula in the lemma. ∎

It is straightforward to check that, for $z \in \mathbf{T}$,

$$\frac{zB'_\#(z)}{B_\#(z)} = \sum_{|\alpha_j|<1}\frac{1-|\alpha_j|^2}{|z-\alpha_j|^2}+\sum_{|\alpha_j|>1}\frac{|\alpha_j|^2-1}{|z-\alpha_j|^2}.$$

Using this relation and the fact that, for $z \in \mathbf{T}$ and $\zeta \notin \mathbf{T}$,

$$\frac{1}{2\pi}\int_{\mathbf{T}} \frac{|dz|}{|z-\alpha|^2} = \frac{1}{|1-|\alpha|^2|},$$

we can obtain the following lemma.

LEMMA 5. *Let $B_{\#}(z)$ be defined as in Lemma 4. Then*

$$\int_{\mathbf{T}} |B'_{\#}(z)|\,|dz| = 2\pi n.$$

3. A BERNSTEIN TYPE INEQUALITY USING INTEGRAL NORM

THEOREM 6. *If $r \in \mathsf{R}_n$ then*

$$\int_{\mathbf{T}} |r'(z)|\,|dz| \leq \int_{\mathbf{T}} |r(z)B'_{\#}(z)|\,|dz|.$$

This theorem is a special case of the following result.

THEOREM 7. *Let $\chi(u)$ be non-decreasing, non-negative, and convex for $u \geq 0$. If $r \in \mathsf{R}_n$, then*

$$\int_{\mathbf{T}} \chi\left(\frac{|r'(z)|}{|B'_{\#}(z)|}\right)|B'_{\#}(z)|\,|dz| \leq \int_{\mathbf{T}} \chi(|r(z)|)|B'_{\#}(z)|\,|dz|. \tag{8}$$

Letting $\alpha_j = 0$ for all $j = 1, 2, \ldots, n$ in Theorem 7 will give us the following Bernstein inequality for polynomials as proved by Zygmund [Z, Chap. X, Theorem 3.16].

COROLLARY 8. *Let $\chi(u)$ be non-decreasing, non-negative, and convex for $u \geq 0$. If $p \in \mathsf{P}_n$, then*

$$\int_{\mathbf{T}} \chi(n^{-1}|p'(z)|)|dz| \leq \int_{\mathbf{T}} \chi(|p(z)|)|dz|.$$

Remark. Equality holds in (8) if r is a constant times $B_{\#}$; this could happen if and only if either all α_j lie inside $\mathbf{T}$ or all α_j lie outside $\mathbf{T}$.

Proof of Theorem 7. Let $\zeta := e^{i\theta}$ and

$$\rho(\theta; z) := \frac{1}{2\pi|B_{\#}'(z)|}\left\{\left|\frac{1 - B_-(z)\overline{B_-(\zeta)}}{1 - z\bar{\zeta}}\right|^2 + \left|\frac{1 - B_+(z)\overline{B_+(\zeta)}}{1 - z\bar{\zeta}}\right|^2\right\}.$$

Then $\rho(\theta; z)d\theta$, with $z \in \mathbf{T}$, is a probability measure on $[0, 2\pi]$ by Lemma 4. From Lemmas 3 and 4, we have, for $r \in \mathcal{R}_n$ and $z \in \mathbf{T}$,

$$|r'(z)| \le |B_{\#}'(z)| \int_0^{2\pi} |r(e^{i\theta})|\, \rho(\theta; z)\, d\theta.$$

Dividing both sides by $|B_{\#}'(z)|$ and applying Jensen's inequality, we get

$$\chi\left(\frac{|r'(z)|}{|B_{\#}'(z)|}\right) \le \int_0^{2\pi} \chi\big(|r(e^{i\theta})|\big)\rho(\theta; z)\, d\theta.$$

On multiplying both sides by $|B_{\#}'(z)|$ and then integrating with respect to z on $\mathbf{T}$, we have

$$\int_{\mathbf{T}} \chi\left(\frac{|r'(z)|}{|B_{\#}'(z)|}\right)|B_{\#}'(z)|\,|dz| \le \int_0^{2\pi} \chi\big(|r(e^{i\theta})|\big)\left\{\int_{\mathbf{T}} |B_{\#}'(z)|\, \rho(\theta; z)|dz|\right\} d\theta.$$

Notice that the integral in the curly brackets equals $|B_{\#}'(e^{i\theta})|$. Now, inequality (8) follows. ∎

4. A NEW PROOF OF THE INEQUALITY OF SPIJKER

We restate the inequality (3) here. Spijker's proof concerned mainly the inequality itself; as for the equality, it mentioned $r(z) = z^n$. Our proof will make it easy to see when the equality holds.

THEOREM 9 [S, Lemma 4]. *If r is a rational function of degree at most n with poles off the unit circle, then*

$$\int_{\mathbf{T}} |r'(z)|\, |dz| \le 2\pi n \|r\|. \tag{9}$$

Furthermore, the equality holds when and only when $r(z)$ is a constant times a Blaschke product of exact degree n and its poles are either all inside $\mathbf{T}$ *or all outside* $\mathbf{T}$.

Proof. The rational function $r(z)$ of degree at most n must belong to some $\mathcal{R}_n$ for some α_j's. Let $B_{\#}(z)$ be associated with α_j's and $\mathcal{R}_n$ as in Lemma 4.

Now, the inequality follows from the facts that

$$\int_{\mathbf{T}} |B'_{\#}(z)|\,|dz| \le 2\pi n \qquad \text{(by Lemma 5)} \tag{10}$$

and

$$\int_{\mathbf{T}} |r'(z)|\,|dz| \le \|r\| \int_{\mathbf{T}} |B'_{\#}(z)|\,|dz|, \tag{11}$$

where the last inequality can be easily obtained from either Theorem 6 or Theorem 1.

The equality holds in (9) if and only if the equalities hold in both (10) and (11). But (10) becomes an equality if and only if $B_{\#}(z)$ (and therefore $r(z)$ itself) is of exact degree n; while (11) renders an equality if and only if, for all $z \in \mathbf{T}$, $|r(z)| = \|r\|$ and

$$\int_{\mathbf{T}} \left\{ \left| \frac{1 - B_{-}(z)\overline{B_{-}(\zeta)}}{1 - z\bar{\zeta}} \right|^2 + \left| \frac{1 - B_{+}(z)\overline{B_{+}(\zeta)}}{1 - z\bar{\zeta}} \right|^2 \right\} |d\zeta|$$

$$= \int_{\mathbf{T}} \left| \left[\frac{1 - B_{-}(z)\overline{B_{-}(\zeta)}}{1 - z\bar{\zeta}} \right]^2 - \left[\frac{1 - B_{+}(z)\overline{B_{+}(\zeta)}}{1 - z\bar{\zeta}} \right]^2 \right| |d\zeta|.$$

Therefore $r(z)$ is of the form const.$\times\, B_n(z)$, and for every $z \in \mathbf{T}$ there exists a constant $k(z) \in [0, \infty]$ such that

$$\left[\frac{1 - B_{-}(z)\overline{B_{-}(\zeta)}}{1 - z\bar{\zeta}} \right]^2 = k(z) \left[\frac{1 - B_{+}(z)\overline{B_{+}(\zeta)}}{1 - z\bar{\zeta}} \right]^2$$

for all $\zeta \in \mathbf{T}$. By comparing the poles, we conclude that the above equality is possible if and only if either $B_{-}(z) \equiv 0$ or $B_{+}(z) \equiv 0$. Combining everything above yields the conclusions on the case for equality in (9). ∎

REFERENCES

[BE] P. Borwein and T. Erdélyi, Sharp extensions of Bernstein's inequality to rational spaces, *Mathematika* **43** (1996), 413–423.

[JLMR] R. Jones, X. Li, R. N. Mohapatra, and R. S. Rodriguez, On the Bernstein inequality for rational functions with a prescribed zero, 1995.

[LMR] X. Li, R. N. Mohapatra, and R. S. Rodriguez, Bernstein-type inequalities for rational functions with prescribed poles, *J. London Math. Soc.* (2) **51** (1995), 523–531.

[LT] R. J. LeVeque and L. N. Trefethen, On the resolvent condition in the Kreiss matrix theorem, *BIT* **24** (1984), 584–591.

[PP] P. P. Petrushev and V. A. Popov, Rational approximation of real functions, *in* ‘‘Encyclopedia of Mathematics and Its Applications,’’ Vol. 28, Cambridge Univ. Press, Cambridge, 1987.

[S] M. N. Spijker, On a conjecture by LeVeque and Trefethen related to the Kreiss matrix theorem, *BIT* **31** (1991), 551–555.

[WT] E. Wegert and L. N. Trefethen, From the Buffon needle to the Kreiss matrix theorem, *Amer. Math. Monthly* **101** (1994), 132–139.

[Z] A. Zygmund, ‘‘Trigonometric Series,’’ 2nd ed., Cambridge Univ. Press, Cambridge, 1959.